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SOME ASPECTS REGARDING THE EQUIVALENCE OF THE EULER AND BRYAN ROTATION SCHEMATA

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Abstract: *In our paper we will make a presentation of the Euler and Bryan rotational schemata proving their equivalence modulo* 2π *for each rotational angle. Reducing the interval of variation to* $[0,2\pi]$*, for each angle for a given rotational schema, there result two different possibilities of rotation for the other schema. Imposing initial conditions, the result is unique, no matter what rotational schema is the original and which one is the transformation. For each possibility, the derivatives of the angles with respect to time are completely determined. A numerical application completes the theory.*

Keywords: Euler, Bryan, equivalence, rotation

INTRODUCTION

In the study of the dynamics of rigid bodies by the multibody type approach, one uses different rotational schemata, the most known of them being Euler's and Bryan's schemata.
It is proved [1] that the possible rotational schemata are only 12; they can be divided into two categories: aba and abc, where a, b and c signifies the axis about which the rotation is performed.
Assigning the values 1, 2, and 3 for the axes x, y, and z, respectively, it is clear that the Euler schema may be denoted by 313, while the Bryan schema may be noted as 123.
Let ψ, θ and φ be the three rotational angles and let $[\psi]$, $[\theta]$, $[\varphi]$ be the corresponding matrices

$$[\boldsymbol{\psi}_E]=\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix},\ [\boldsymbol{\theta}_E]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix},\ [\boldsymbol{\varphi}_E]=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{1}$$

$$[\boldsymbol{\psi}_B]=\begin{bmatrix}1 & 0 & 0\\ \cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\end{bmatrix},\ [\boldsymbol{\theta}_B]=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix},\ [\boldsymbol{\varphi}_B]=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{2}$$

In the previous relations the indices E, and B stay for Euler, and Bryan, respectively.
It results the rotational matrices

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$$[\mathbf{A}_E] = [\boldsymbol{\psi}_E][\boldsymbol{\theta}_E][\boldsymbol{\varphi}_E] = \begin{bmatrix} c\psi c\varphi - s\psi c\theta s\varphi & -c\psi s\varphi - s\psi c\theta c\varphi & s\psi s\theta \\ s\psi c\varphi + c\psi c\theta s\varphi & -s\psi s\varphi + c\psi c\theta c\varphi & -c\psi s\theta \\ s\theta s\varphi & s\theta c\varphi & c\theta \end{bmatrix}, \quad (3)$$

$$[\mathbf{A}_B] = [\boldsymbol{\psi}_B][\boldsymbol{\theta}_B][\boldsymbol{\varphi}_B] = \begin{bmatrix} c\theta c\varphi & -c\theta s\varphi & s\theta \\ s\psi s\theta c\varphi + c\psi s\varphi & -s\psi s\theta s\varphi + c\psi c\varphi & -s\psi c\theta \\ -c\psi s\theta c\varphi + s\psi s\varphi & c\psi s\theta s\varphi + s\psi c\varphi & c\psi c\theta \end{bmatrix}. \quad (4)$$

Let $[\mathbf{Q}]$ be the matrix defined by

$$[\mathbf{Q}] = [\boldsymbol{\varphi}]^{\mathrm{T}}\left[[\boldsymbol{\theta}]^{\mathrm{T}}\{\mathbf{u}_\psi\} \;\; \{\mathbf{u}_\theta\} \;\; \{\mathbf{u}_\varphi\}\right], \quad (5)$$

where $\{\mathbf{u}_\alpha\}$ has the values

$$\{\mathbf{u}_\psi\} = [1 \;\; 0 \;\; 0]^{\mathrm{T}}, \; \{\mathbf{u}_\theta\} = [0 \;\; 1 \;\; 0]^{\mathrm{T}}, \; \{\mathbf{u}_\varphi\} = [0 \;\; 0 \;\; 1]^{\mathrm{T}}, \quad (6)$$

depending on the axis about which the rotation of angle α takes place: x, y or z.
One obtains

$$[\mathbf{Q}_E] = \begin{bmatrix} \sin\varphi\sin\theta & \cos\varphi & 0 \\ \cos\varphi\sin\theta & -\sin\varphi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}, \quad (7)$$

$$[\mathbf{Q}_B] = \begin{bmatrix} \cos\varphi\cos\theta & \sin\varphi & 0 \\ -\sin\varphi\cos\theta & \cos\varphi & 0 \\ \sin\theta & 0 & 1 \end{bmatrix}. \quad (8)$$

The matrix of angular velocities reads

$$\{\boldsymbol{\omega}\} = [\mathbf{Q}]\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix}. \quad (9)$$

One also deduces the matrix of angular accelerations as

$$\{\boldsymbol{\varepsilon}\} = \{\dot{\boldsymbol{\omega}}\} = [\dot{\mathbf{Q}}]\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} + [\mathbf{Q}]\begin{bmatrix} \ddot{\psi} \\ \ddot{\theta} \\ \ddot{\varphi} \end{bmatrix}, \quad (10)$$

where

$$[\dot{\mathbf{Q}}] = [\dot{\boldsymbol{\varphi}}]^{\mathrm{T}}\left[[\boldsymbol{\theta}]^{\mathrm{T}}\{\mathbf{u}_\psi\} \;\; \{\mathbf{u}_\theta\} \;\; \{\mathbf{u}_\varphi\}\right] + [\boldsymbol{\varphi}]^{\mathrm{T}}\left[[\dot{\boldsymbol{\theta}}]^{\mathrm{T}}\{\mathbf{0}\} \;\; \{\mathbf{0}\} \;\; \{\mathbf{0}\}\right], \quad (11)$$

in which

$$\{\mathbf{0}\} = [0 \;\; 0 \;\; 0]^{\mathrm{T}}. \quad (12)$$

DETERMINATION OF THE BRYAN ROTATIONAL ANGLES KNOWING THE EULER ROTATIONAL ANGLES

Knowing the Euler rotational angles ψ_E, θ_E, and φ_E, we may determine the rotational matrix in the form

$$[\mathbf{A}_E] = [\boldsymbol{\psi}_E][\boldsymbol{\theta}_E][\boldsymbol{\varphi}_E] = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}. \tag{13}$$

Equating the rotational matrices $[\mathbf{A}_E]$ and $[\mathbf{A}_B]$ ($[\mathbf{A}_E] = [\mathbf{A}_B]$), the element situated on the first row and third column in the matrix $[\mathbf{A}_B]$ offers

$$\sin\theta = \alpha_3, \tag{14}$$

wherefrom

$$\theta_1 = \arcsin\alpha_3,\ \theta_2 = \pi - \arcsin\alpha_3, \tag{15}$$

that is, there are two possibilities for the angle θ in the interval $[0,2\pi]$.
In addition

$$\cos\theta_1 = \cos(\arcsin\alpha_3),\ \cos\theta_2 = -\cos(\arcsin\alpha_3). \tag{16}$$

Examining the first line in the matrix $[\mathbf{A}_B]$, we get

$$\cos\varphi = \frac{\alpha_1}{\cos\theta},\ \sin\varphi = -\frac{\alpha_2}{\cos\theta}; \tag{17}$$

hence, the angle φ_3 is determined (two possible values in the interval $[0,2\pi]$).
The third column of the matrix $[\mathbf{A}_B]$ gives the values

$$\sin\psi = -\frac{\beta_3}{\cos\theta},\ \cos\psi = \frac{\gamma_3}{\cos\theta} \tag{18}$$

and, consequently, the angle ψ_B.

One may observe that the problem has not a unique solution. The expressions (15) are written only for $\theta \in [0,2\pi]$. Generally, we discuss about a double infinity of solutions given by

$$\theta = \arcsin\alpha_3 + 2k\pi,\ \theta = (2k+1)\pi - \arcsin\alpha_3, \tag{19}$$

where $k \in \mathbf{Z}$.

DETERMINATION OF THE EULER ROTATIONAL ANGLES KNOWING THE BRYAN ROTATIONAL ANGLES

The problem is completely similar and let $[\mathbf{A}_B]$ be the rotational matrix

$$[\mathbf{A}_B] = [\boldsymbol{\psi}_B][\boldsymbol{\theta}_B][\boldsymbol{\varphi}_B] = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}. \tag{20}$$

The element situated on the third row and third column in the matrix $[\mathbf{A}_E]$ leads to

$$\cos\theta = \gamma_3, \tag{21}$$

wherefrom

$$\theta_1 = \arccos \gamma_3 , \ \theta_2 = 2\pi - \arccos \gamma_3 . \tag{22}$$

Examining now the last line and the last column in the matrix $[\mathbf{A}_E]$, we get

$$\sin \varphi = \frac{\gamma_1}{\sin \theta} , \ \cos \varphi = \frac{\gamma_2}{\sin \theta} , \tag{23}$$

$$\sin \psi = \frac{\alpha_3}{\sin \theta} , \ \cos \psi = -\frac{\beta_3}{\sin \theta} , \tag{24}$$

where

$$\sin \theta_1 = \sin(\arccos \gamma_3) , \ \sin \theta_2 = -\sin(\arccos \gamma_3) . \tag{25}$$

In conclusion, the angles φ_E and ψ_E are also determined.
Some remarks must be made:
- relations (17), (18), (23) and (24) do not offer a unique solution for the angles φ and ψ. For instance, if we determine a solution φ^* of the equation (17), the general solution is

$$\varphi = \varphi^* + 2k\pi , \tag{26}$$

where k is an arbitrary integer;
- the problem has a simpler solution if $\alpha_3 = \pm 1$ (in the case of the matrix $[\mathbf{A}_E]$) or $\gamma_3 = \pm 1$ (in the case of the matrix $[\mathbf{A}_B]$). We will not discuss here the singularities of the dynamical equations of motion.

THE ANGULAR VELOCITY MATRIX

Knowing the Euler angles and their variations, we may write

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [\mathbf{Q}_E] \begin{bmatrix} \dot{\psi}_E \\ \dot{\theta}_E \\ \dot{\varphi}_E \end{bmatrix} . \tag{27}$$

For the Bryan angles we have

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [\mathbf{Q}_B] \begin{bmatrix} \dot{\psi}_B \\ \dot{\theta}_B \\ \dot{\varphi}_B \end{bmatrix} . \tag{28}$$

Equating the last two relations, we obtain

$$[\mathbf{Q}_E] \begin{bmatrix} \dot{\psi}_E \\ \dot{\theta}_E \\ \dot{\varphi}_E \end{bmatrix} = [\mathbf{Q}_B] \begin{bmatrix} \dot{\psi}_B \\ \dot{\theta}_B \\ \dot{\varphi}_B \end{bmatrix} . \tag{29}$$

Assuming now that $[\mathbf{Q}_E]$ and $[\mathbf{Q}_B]$ are invertible, we may write

$$\begin{bmatrix}\dot{\psi}_E\\ \dot{\theta}_E\\ \dot{\varphi}_E\end{bmatrix} = [\mathbf{Q}_E]^{-1}[\mathbf{Q}_B]\begin{bmatrix}\dot{\psi}_B\\ \dot{\theta}_B\\ \dot{\varphi}_B\end{bmatrix}, \tag{30}$$

$$\begin{bmatrix}\dot{\psi}_B\\ \dot{\theta}_B\\ \dot{\varphi}_B\end{bmatrix} = [\mathbf{Q}_B]^{-1}[\mathbf{Q}_E]\begin{bmatrix}\dot{\psi}_E\\ \dot{\theta}_E\\ \dot{\varphi}_E\end{bmatrix}. \tag{31}$$

Let us observe that

$$\det[\mathbf{Q}_E] = -\sin\theta_E, \tag{32}$$

$$\det[\mathbf{Q}_B] = \cos\theta_B \tag{33}$$

and the singularities appear for $\theta_E = k\pi$, $\theta_B = \frac{\pi}{2} + k\pi$, $k \in \mathbf{Z}$, that is, they appear when $\alpha_3 = \pm 1$ or $\gamma_3 = \pm 1$ (see also the previous paragraph).

NUMERICAL EXAMPLE

Let us assume the Euler rotational schema for which

$$\psi_E = 30 + t - 1,\ \theta_E = 45 + t^2 - 1,\ \varphi_E = 60 + t^3 - t^2, \tag{34}$$

where the angles are given in degrees, while t is the time.
For $t = 1$ determine:
i) the matrix of rotation $[\mathbf{A}_E]$;
ii) the corresponding rotational angles for the Bryan schema;
iii) the matrix of angular velocities $\{\boldsymbol{\omega}_E\}$;
iv) the matrix $[\dot{\psi}_B\ \ \dot{\theta}_B\ \ \dot{\varphi}_B]^{\mathrm{T}}$.
Solution: i) We may successively write

$$\psi_E(t = 1) = 30^0,\ \theta_E(t = 1) = 45^0,\ \varphi_E(t = 1) = 60^0, \tag{35}$$

$$\dot{\psi}_E(t = 1) = 1,\ \dot{\theta}_E(t = 1) = 2,\ \dot{\varphi}_E(t = 1) = 1, \tag{36}$$

$$[\mathbf{A}_E] = \begin{bmatrix}0.12683 & -0.92678 & 0.35355\\ 0.78033 & -0.12683 & -0.61237\\ 0.61237 & 0.35355 & 0.70711\end{bmatrix}. \tag{37}$$

ii) From the relation

$$\sin\theta_B = 0.35355 \tag{38}$$

we deduce (for $\theta_B \in [0,2\pi]$)

$$\theta_{B_1} = 20.70460^0 = 0.36136\,\mathrm{rad},\ \theta_{B_2} = 159.29540^0 = 2.78023\,\mathrm{rad}. \tag{39}$$

Further on, we consider

$$\theta_B = 20.70460^0 = 0.36136\,\mathrm{rad} \tag{40}$$

and we have

$$\sin\psi_B = -\frac{-0.61237}{\cos 20.70460^0} = 0.65465,\ \cos\psi_B = \frac{0.70711}{\cos 20.70460^0} = 0.75593,$$
$$\psi_B = 40.89312^0 = 0.71372\,\text{rad}, \tag{41}$$

$$\cos\varphi_B = \frac{0.12683}{\cos 20.70460^0} = 0.13559,\ \sin\varphi_B = -\frac{-0.92678}{\cos 20.70460^0} = 0.99077,$$
$$\varphi_B = 82.20745^0 = 1.43479\,\text{rad}. \tag{42}$$

The reader may observe that we have limited to the interval $[0,2\pi]$.

iii) We write

$$[\mathbf{Q}_E] = \begin{bmatrix} 0.61237 & 0.50000 & 0 \\ 0.35355 & -0.86603 & 0 \\ 0.70711 & 0 & 1 \end{bmatrix} \tag{43}$$

and it results

$$[\boldsymbol{\omega}_E] = \begin{bmatrix} 0.61237 & 0.50000 & 0 \\ 0.35355 & -0.86603 & 0 \\ 0.70711 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.61237 \\ -1.37851 \\ 1.70711 \end{bmatrix}. \tag{44}$$

iv) We calculate

$$[\mathbf{Q}_B] = \begin{bmatrix} 0.12683 & 0.99077 & 0 \\ -0.35029 & 0.13559 & 0 \\ 0.35355 & 0 & 1 \end{bmatrix}, \tag{45}$$

$$[\mathbf{Q}_B]^{-1} = \begin{bmatrix} 0.36618 & -2.67573 & 0 \\ 0.94601 & 0.34252 & 0 \\ -0.12947 & 0.94601 & 1 \end{bmatrix}, \tag{46}$$

$$\begin{bmatrix} \dot{\psi}_B \\ \dot{\theta}_B \\ \dot{\varphi}_B \end{bmatrix} = [\mathbf{Q}_B]^{-1}\{\boldsymbol{\omega}_E\} = \begin{bmatrix} 4.27894 \\ 1.05315 \\ 0.19427 \end{bmatrix}. \tag{47}$$

CONCLUSIONS

This paper shows the way in which one may pass from a rotational schema to another. The results may be easily generalized for two arbitrary rotational schemata, one of them being of *aba* type and the one of *abc* type. In this way, the singularities that appear in the equations of motion, reported in [1] and [2] may be avoided. The passing formulae are not uniquely determined, but considering the initial conditions they become unique, so the transformation is a one-to-one transformation.

REFERENCES

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